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Letter to the Editor

On the exponential stability of a class of nonlinear systems including delayed perturbations

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Abstract

In this letter, the problem of robust stabilization of a class of nonlinear dynamical systems with delayed perturbations is considered. Based on the stability of the nominal systems, a new stabilizing control law for exponential stability of the system is designed using Lyapunov stability theory.

Keywords: Nonlinear system; Exponential stability; Delayed perturbations; Lyapunov function

1. Introduction

The problem of robust stabilization of uncertain dynamical systems has received considerable attention over the decades, since perturbations which cause instability are encountered in various systems. Regarding the system with perturbations, many researchers have devoted themselves to develop effective control techniques to guarantee the stability. Since the two remarkable works, which are the discontinuous minmax control [6] and continuous saturation-type control [3], have been introduced, numerous reports have been published on the controller design for stability of uncertain systems [1,4,5,9]. On the other hand, if the system has delayed perturbations, the stability analysis becomes much more difficult job. In the literature, the problem of robust stability and stabilization of linear time-invariant systems with delayed perturbation has been studied [2,7,8,10].

In this article, we consider the problem of robust stabilization for a class of nonlinear dynamic systems subject to nonlinear delayed perturbations described by

$$\dot{x}(t) = F(x(t),t) + G(x,t)[H(x(t-h(t)),t) + u(t)], \tag{1}$$

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where $t \in \mathscr{R}$ is time, $x(t) \in \mathscr{R}^n$ is the state vector, $u(t) \in \mathscr{R}^m$ is the control vector, and system perturbation $H(x(t-h(t)),t) \in \mathscr{R}^n$ is a time-varying nonlinear continuous function of the state, $H(0,t)=0\ \forall t$ and is assumed to be bounded in magnitude, usually in its Euclidean norm. The time delay $h(t)$ is any nonnegative, bounded, and continuous function. That is, $0 \leqslant h(t) \leqslant \bar{h}$, where $\bar{h}$ is any constant. The initial condition function is given by $x(t) = \phi(t)$, $t \in [-\bar{h}, 0]$, where $\phi(t)$ is a continuous vector-valued initial function on $[-\bar{h}, 0]$. The term $G(\cdot)H(\cdot)$ represents the matching condition [1–5,7,10] about the perturbation. It should be noted that the matching condition is often not satisfied in some applications.

The corresponding system without perturbations, called nominal system, is described by

$$\dot{x}(t) = F(x,t) + G(x,t)u(t), \tag{2}$$

where $F(\cdot)$: $\mathscr{R}^n \times \mathscr{R} \to \mathscr{R}^n$ is known and stable and $G(\cdot)$: $\mathscr{R}^n \times \mathscr{R} \to \mathscr{R}^{n\times m}$ is known.

Here, the goal of the letter is to design a new continuous state-feedback control law that allow system (1) exponential stability in the presence of delayed perturbations $H(\cdot)$ using Lyapunov functional theory. Throughout the letter, $\|\cdot\|$ refers to either the Euclidean vector norm or the induced matrix 2-norm. $\lambda_m(\cdot)$ and $\lambda_M(\cdot)$ denote minimum and maximum eigenvalue of the matrix $(\cdot)$, respectively. $|\cdot|$ denotes absolute value.

2. Main result

Before giving our main result, the following assumptions are introduced for control design.

Assumption 2.1. For mathematical completeness, the known function $F(\cdot), G(\cdot)$ and the unknown function $H(\cdot)$ are continuous, uniformly bounded with respect to time, and locally uniformly bounded with respect to the state.

Assumption 2.2. The perturbation $H(\cdot)$ is bounded in Euclidean norm as follows:

$$\|H(x(t-h(t)),t)\| \leqslant \beta\|x(t-h(t))\|, \quad \beta > 0. \tag{3}$$

Assumption 2.3. The origin $x = 0$ is a uniformly exponentially stable equilibrium point of nominal system (2). Also, there exists a C^1 function $V(\cdot,\cdot)$: $\mathscr{R}^n \times \mathscr{R} \to \mathscr{R}^+$ which satisfies

$$(\lambda_1\|x\|)^2 \leqslant V(x,t) \leqslant (\lambda_2\|x\|)^2, \tag{4}$$

$$\frac{\partial V(x,t)}{\partial t} + \nabla_x^{\mathrm{T}} V(x,t)F(x,t) \leqslant -\lambda_3 V(x,t) \tag{5}$$

for all $(x,t) \in \mathscr{R}^n \times \mathscr{R}$, where λ_1, λ_2, and λ_3 are positive scalars.

Now, we propose the following state feedback control law:

$$u(t) = -\frac{G^{\mathrm{T}}(x,t)\nabla_x V(x,t)\beta^2\chi^2(t)}{\|\nabla_x^{\mathrm{T}} V(x,t)G(x,t)\|\beta\chi(t) + \varepsilon \mathrm{e}^{-\alpha t}}, \tag{6}$$

where the design parameters ε and α are positive scalars and

$$\chi(t) = \max\|x(t)\| \quad \forall t \in [t - \bar{h}, t]. \tag{7}$$

Then we state the stability behavior of the closed-loop systems formed by (1) and (6). The following theorem provides our stability result concerning the control law (6).

Theorem 2.1. *Consider dynamical system* (1) *satisfying Assumptions* 2.1–2.3. *Then, the system is exponentially stable in the sense of*

$$\|x(t)\| \leqslant \begin{cases} [(\frac{\lambda_2}{\lambda_1}\|x(0)\|)^2 e^{-\lambda_3 t} + \frac{\varepsilon}{\lambda_1^2} t e^{-\lambda_3 t}]^{1/2} & \text{if } \lambda_3 = \alpha, \\ [(\frac{\lambda_2}{\lambda_1}\|x(0)\|)^2 e^{-\lambda_3 t} + \frac{\varepsilon}{\lambda_1^2(\lambda_3 - \alpha)}[e^{-\alpha t} - e^{-\lambda_3 t}]]^{1/2} & \text{if } \lambda_3 \neq \alpha, \end{cases} \tag{8}$$

where the positive scalars ε and α are given in (6).

Proof. Using the same Lyapunov function as given in Assumption 2.3, we have

$$\begin{aligned} \dot{V}(x,t) &= \frac{\partial V(x,t)}{\partial t} + \nabla_x^T V(x,t)\dot{x}(t) \\ &= \frac{\partial V(x,t)}{\partial t} + \nabla_x^T V(x,t)[F(x,t) + G(x,t)[H(x(t-h(t)),t) + u(t)]] \\ &\leqslant -\lambda_3 V(x,t) + \|\nabla_x^T V(x,t)G(x,t)\| \cdot \|H(x(t-h(t)),t)\| + \nabla_x^T V(x,t)G(x,t)u(t). \end{aligned} \tag{9}$$

Substituting (3) and (6) into (9) yields

$$\begin{aligned} \dot{V}(x,t) &\leqslant -\lambda_3 V(x,t) + \|\nabla_x^T V(x,t)G(x,t)\| \cdot \beta \cdot \|x(t-h(t))\| - \frac{\|\nabla_x^T V(x,t)G(x,t)\|^2 \beta^2 \chi(t)^2}{\|\nabla_x^T V(x,t)G(x,t)\|\beta\chi(t) + \varepsilon e^{-\alpha t}} \\ &\leqslant -\lambda_3 V(x,t) + \|\nabla_x^T V(x,t)G(x,t)\|\beta\chi(t) - \frac{\|\nabla_x^T V(x,t)G(x,t)\|^2 \beta^2 \chi(t)^2}{\|\nabla_x^T V(x,t)G(x,t)\|\beta\chi(t) + \varepsilon e^{-\alpha t}} \\ &= -\lambda_3 V(x,t) + \rho(t) - \frac{\rho^2(t)}{\rho(t) + \varepsilon e^{-\alpha t}} = -\lambda_3 V(x,t) + \frac{\rho(t)\varepsilon e^{-\alpha t}}{\rho(t) + \varepsilon e^{-\alpha t}}, \end{aligned} \tag{10}$$

where $\rho(t) = \|\nabla_x^T V(x,t)G(x,t)\|\beta\chi(t)$.

Therefore, it follows from (10) and from the inequality $0 \leqslant ab/(a+b) \leqslant a \ \forall a, b > 0$ that

$$\dot{V}(x,t) \leqslant -\lambda_3 V(x,t) + \varepsilon e^{-\alpha t}. \tag{11}$$

By utilizing the result of [5] to (11) inequality (8) can be easily obtained which completes the proof. □

Remark 2.1. An exponential function $\varepsilon e^{-\alpha t}$ [5] is included in the control law (6). This is different either from [2,7] where a constant ε is used, or from [9] where a state-dependent function $\varepsilon\|x(t)\|^2$ is employed.

As a special case of above result, consider uncertain systems with linear nominal part described in [2,7,8,10]

$$\dot{x}(t) = Ax(t) + BH(x(t-h(t)),t) + Bu(t), \tag{12}$$

where $A \in \mathscr{R}^{n\times n}$ is a Hurwitz matrix, $B \in \mathscr{R}^{n\times m}$ is a constant matrix, the matrix pairs (A, B) are completely controllable, and $\|H(\cdot)\| \leqslant \beta\|x(t-h(t))\|$.

Here, we define a Lyapunov equation as follows:

$$\tfrac{1}{2}[A^{\mathrm{T}}P + PA] = -Q, \tag{13}$$

where for given positive definite matrix Q, the solution P is a positive definite matrix.

Now, for positive scalars ε and α, we propose the following feedback control law for system (12):

$$u(t) = -\frac{B^{\mathrm{T}}Px(t)\beta^2\chi(t)^2}{\|B^{\mathrm{T}}Px(t)\|\beta\chi(t) + \varepsilon \mathrm{e}^{-\alpha t}}. \tag{14}$$

Then, we have the following corollary.

Corollary 2.1. *Consider system* (12) *satisfying Assumptions* 2.1 *and* 2.2. *Then, the system is exponentially stable in the sense of Eq.* (8) *with* $\lambda_1 = \frac{1}{\sqrt{2}}\sqrt{\lambda_m(P)}$, $\lambda_2 = \frac{1}{\sqrt{2}}\sqrt{\lambda_M(P)}$, $\lambda_3 = 2\lambda_m(Q)/\lambda_M(P)$, *and* $\lambda_4 = \varepsilon$.

Proof. Select the Lyapunov function $V = 0.5x^{\mathrm{T}}(t)Px(t)$, which satisfies $0.5\lambda_m(P)\|x\|^2 \leqslant V \leqslant 0.5\lambda_M(P)\|x\|^2$. Then, it follows that

$$\begin{aligned}\dot{V} &\leqslant -x^{\mathrm{T}}Qx + \|B^{\mathrm{T}}Px\|\beta\|x(t-h(t))\| + x^{\mathrm{T}}PBu \leqslant -\lambda_m(Q)\|x\|^2 + \|B^{\mathrm{T}}Px\|\beta\chi + x^{\mathrm{T}}PBu \\ &\leqslant -\frac{2\lambda_m(Q)}{\lambda_M(P)}V + \|B^{\mathrm{T}}Px\|\beta\chi + x^{\mathrm{T}}PBu. \end{aligned} \tag{15}$$

Using the same manipulation of proof of Theorem 2.1, we have $\dot{V} \leqslant -(2\lambda_m(Q)/\lambda_M(P))V + \varepsilon \mathrm{e}^{-\alpha t}$. The rest of proof is obvious. □

Remark 2.2. In [2,7,8], robust controllers are designed, but state convergence to the origin yields only ultimate boundedness stability or asymptotic stability instead of exponential stability.

Example. Consider system (1) with

$$F(\cdot) = \begin{bmatrix} -x_1 + 3x_1x_2^2 \\ -x_2 - 2x_1^2x_2 \end{bmatrix}, \quad G(\cdot) = \begin{bmatrix} x_1x_2 & 0 \\ 0 & 2x_2 \end{bmatrix}, \quad H(\cdot) = \begin{bmatrix} 1.2\sin(x_1(t-h(t))) \\ 1.6\sqrt{x_1(t-h(t))x_2(t-h(t))} \end{bmatrix}.$$

In order to construct the robust control law given in (6) which gives exponential stability of the system, let us define a Lyapunov function $V(x,t)$ for the nominal system of above system. We choose a quadratic function of the form

$$V(x,t) = x^{\mathrm{T}}(t)\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}x(t).$$

Then, we can determine the constants $\lambda_1, \lambda_2, \lambda_3$ and β in the light of this Lyapunov function and uncertain function. It is readily shown that $\lambda_1 = \sqrt{2}$, $\lambda_2 = \sqrt{3}$, $\lambda_3 = 2$, $\beta = 2$, i.e.,

$$(\sqrt{2}\|x\|)^2 \leqslant V(x,t) \leqslant (\sqrt{3}\|x\|)^2, \quad \frac{\partial V(x,t)}{\partial t} + \nabla_x^{\mathrm{T}}V(x,t)F(x,t) \leqslant -2 \times V(x,t).$$

Then, in the light of (6), the feedback controller guaranteeing the exponential stability of the above system can be represented by

$$u(t) = -\frac{4\chi^2(t)}{8|x_2|\sqrt{x_1^4 + 9x_2^2}\chi(t) + 0.5e^{-t}} \begin{bmatrix} 4x_1^2 x_2 \\ 12x_2^2 \end{bmatrix}, \tag{16}$$

where $\chi(t)$ is given by (7), and the control parameter ε and α is chosen as $\varepsilon = 0.5$ and $\alpha = 1$.

Moreover, we obtain an estimate of the convergence for the exponential stability of the closed-loop system as $\|x(t)\| \leqslant [0.25e^{-t} + 3.5e^{-2t}]^{1/2}$ $\forall t > 0$.

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